Projections And Intersection Properties Of Fractals

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Introduction

The term "fractals" refers to an unbounded pattern of composite geometric objects that provide a means of visualizing and measurably observing a range of analogous conceptual and real-world occurrences across diverse scales in an iterative manner (Falconer, 2013). They are very helpful in the modeling of structures that have similar or progressively recurring small-scale patterns. A good example of such structures is the snowflakes (Di Leva, 2016). They are also used to describe random patterns such as galaxy development or crystal growth. The first person to ever use fractals was Benoit Mandelbrot in 1975 (Falconer, 2013). He based his definition of fractals on the Latin word which meant "broken" or fractured.

This paper explains that fractal sets are subsets of Rn (n = 1,2,3 here) which have fractional (i.e., non-integer) dimensions. This project will look at some 'strange' properties of fractals, more so the projection and intersection properties of fractals. One of the results to be discussed is the existence of a 'digital sundial'; that is, there exists a subset E in R3 such that the shadow cast by E (i.e., its projection onto the ground), when the sun shines on it, gives a digital reading of the time, and as the sun moves the shadow changes to continue giving the correct time. We also look at sets with 'large intersections.' These are sets that can have very small dimensions, so they are 'very small' but are 'spread out' in such a way that when they are intersected, even infinitely often, their dimension does not shrink (when curves or surfaces have intersected the dimension of the intersection tends to be smaller than the original dimensions). These large intersection sets also have the bizarre property of being 'very small' in a measure theory sense but 'large' in a topological sense.

There are several real-world fractal examples, such as the coastline of Great Britain, which has a jagged edge, and the leaves of plants, to mention a few. In mathematical terms, a fractal is defined as a set consisting of R k with a jagged-shaped structure. In order to clearly determine R k, some terms used need to be clarified to define what qualifies as a jagged structure.

The Set Theory:

These are real numbers denoted by R for an integer k >_ 1

R k = {x = (x_1, . . . , x_k): xi ∈ R, i = 1, . . . , k}.

If the result is $A \subset R_k$, then the A compliment is considered a set

A c: = $\{x \in R_k: x \ 6\in A\}$.

When $B \subset R_k$, then

A B: = $\{x \in A : x \ 6\in B\}$

When f: $R_k \to R_l$ and $B \subset R_{l,}$ then the sets are defined as: f(A) := $\{y \in R_l : y = f(x), x \in A\}$, f −1 (B) := $\{x \in R_k : f(x) \in B\}$.

The point to note is that.

A B = A ∩ B c, f −1 (A c) = (f −1 (A))c.

When $A \subset R$ k, $B \subset R$ l, then the Cartesian product of A and B is the set A × B := $\{(x, y) \in R$ k+l : $x \in A, y \in B\}$.

A set $A \subset R$ k can be counted if its elements of appear in a 'list' such as A = {a1, a2, a3,.

}. For Example, the set of positive integers N = {1, 2, 3, . . . } is countable.

Determining the distance between a set and a point:

To achieve this, there is a theory which states that:

For any points $x \in R$ k and any set $A \subset R$ k,

Let distance (x, A) := $\inf\{|x - y| : y \in \bar{A}\}$.

There is a point $y \in A$ where distance (x, A) = $|x - y|$. Is the shortest distance between point x and y in A.

To further elaborate on the above, the example below would be most appropriate.

Let x = 4 and A = [0, 1], B = (0, 1). Then distance (x, A) = $|4 - 1| = 3$, and also distance (x, B) =

3, even without point $y \in B$ where $|x - y| = 3$ [clearly, $|x - y| > 3$ for all $y \in B$ and for any ǫ > 0 there always is $y \in B$ such that $|x - y| < 3 +$ ǫ so the inf = 3.]

The theorem below is deduced:

For any $x \in R$ k and any $A \subset R$ k, distance (x, A) = distance (x, $\bar{A}$).

Measurable Properties

There is a rule referred to as the Lebesgue measure μ, which considers sets $A \subset R$ k and gives it a specific number $\mu(A) \in [0, \infty]$; this is what is referred to as the Lebesgue's measure of A. When used in R, the measure is more general in terms of intervals and the length under consideration. Whereas in R2, the area is more generalized, and volume is generalized in R3. This makes its construction a bit complicated. The main disadvantage is the complication and the inability to assign measures to all the $A \subset R^k$ sets. Therefore, two sets are realized: the measurable (those sets that can be measured) and the immeasurable (those sets that prove too complicated to assign any measure. It is, however, practically impossible to determine an immeasurable set because all the sets that are encountered are measurable.

Lemma μ Properties

1. All the measurable sets' compliments are measurable. The intersections and the unions that can be counted within the set are all measurable, and all closed and open sets are equally measurable.
2. $\mu(\varnothing) = 0$.
3. Note that set $R = [a_1, b_1] \times . . . \times [a_k, b_k] \subset R^k$ a rectangle. For any rectangle R, $\mu(R) = (b_1 - a_1) \times . . . \times (b_k - a_k)$, that is, μ(R) is the length, area or volume of R in $R/R^2/R^3$.
4. $A \subset B \Longrightarrow \mu(A) >\_ \mu(B)$.

A good example to illustrate the above is:

μ({x}) = 0 for all points x ∈ R k; that is, the measure of the point is zero. Hence, the measure of a countable collection of points is zero (by property (4). Since the set of all rational numbers in R is countable, it has measure zero (NB: the set of irrational numbers is not countable). (2) μ((0, 1)) = μ([0, 1]) = μ((0, 1]) = μ([0, 1)) = 1 (by property (3) and part (1) of this example). (3) μ(Br(x)) = πr2 when n = 2, μ(Br(x)) = 4 3 πr3 when n = 3 (since μ generalizes area and volume when n = 2, 3). The same set (essentially) can have varied measures depending on the space it is regarded as being in. such as. μ1([0, 1]) = 1, but μ2([0, 1] × {0}) = 0.

For the second case, the line segment [0, 1] × {0} on the x-axis in R 2 is regarded as a rectangle with zero height — as a line segment, it has a length of 1 but an area of 0).

For each m > 0: (i) The set E_m is closed and consists of 2_m closed intervals of length 3^{-m}. The lengths of the gaps between the intervals of E_m are at least 3^{-m}.

$\mu(E_m) = 2^{m3\ -m} = (2/ 3)^m$.

For any m0 > 0, $F_C = \cap^{\infty}_{m=m0} E_m$.

The first proof is that their construction is by induction, and it is as follows:

E0 is a single closed interval of length 1.

Now, suppose that the result is true for some j ≥ 1. Then each interval I of E_j gives rise to 2 intervals I_1, I_2 of E_{j+1}, and $|I_1| = |I_2| = {}^1/_3\ |I| = 3-^{(j+1)}$, and the gap in between is I_1, I_2 is $3^{-(j+1);}$ the gap is I_1, I_2 and any other interval in E_{j+1} is at least the gap between the intervals of E_j , i.e., 3^{-j} . Thus the smallest gaps in E^{j+1} are $3^{-(j+1)}$. (ii) This follows immediately from property (i). (iii) Since E0 ⊃ E1 ⊃ E2 ⊃ . . . ,

The Von Koch Curve

Suppose t $E_0 = [0, 1] \times \{0\} \subset R^{2`}$. E_1 can be constructed by removing the central open middle-third interval from E0 and replacing it with the two upper sides of the equilateral triangle, the base of which is the central interval (Falconer, 2013).

Construct E_2 by removing the open middle third interval at the center from every line segment of E_1. And have it replaced by the other two sides of the equilibrium triangle whose base forms the removed equilibrium. By repeating this process, a sequence set of E_0, E_1, E_2, . . . , in R^2 is obtained. Thus, the set is defined as F_K: = limm E_m, / →∞. This is not an easy task since it is not clear what the limit means. It is, therefore, defined as the set F_K that we eventually obtain, which will be called the von Koch curve. Let’s begin with some properties of the sets E_m, similar to those in the past examples.

Example

For each m ≥0 : (i) The set E_m is closed and consists of 4m line segments, each of length 3^{-m}. (ii) The total length of E^m is $4^m 3^{-m} = (4/3)^m$

Proof: To get from E_m to E_{m+1} we replace each line segment in Em, of length 3^{-m}, with 4 line segments, of length $3^{-(m+1).}$ Thus (i) follows by induction, and (ii) follows from (i). addition

10 It follows from Lemma 2.5 that even if we can regard the limit set F^K as a curve at all, it will be a very ‘jagged’ curve, and will have infinite length. Of course, it is not yet clear that the limit will even be a curve. To define the limit in (2.3), one can construct functions f_m: [0, 1] → R^2 such that the set E_m = f_m ([0, 1]) and then define the limit in (2.3) in terms of a limit of these functions. To do this, we first need to define the limit of a sequence of functions, and to do this, we need to define a distance between functions.

For any continuous functions f : [0, 1] → R 2 , g : [0, 1] → R^2 , we define the distance between f, g as ‖f − k‖ := sup{| f(x) – g (x)| : x ∈. [0, 1]}. The following broad outcomes about the convergence of a series of functions are noted:

If f_m : [0, 1] → R^k , m = 1, 2, . . . , is a sequence of continuous functions and if X∞ m=1 kfm − fm+1k < ∞, then there exists a continuous function f∞ : [0, 1] → R k such that, as m → ∞, kfm − f∞k → 0.

(2.4) Also, for fixed m > 1, kfm − fjk → kfm − f∞k

Hence, X∞ m=1 kfm − fm+1k < X∞ m=1 3 −(m+1) < ∞, and it shows that there exists a continuous function f∞ : [0, 1] → R 2 such that kfm − f∞k → 0. (2.8) We now define FK := f∞([0, 1]).

The above set has the following properties.

1. $F_K \neq \emptyset$; in fact, F_K contains all the corners in the set Em, for all m > 0.
2. F_K is closed.
3. $\| f_m - f\infty k \| \leq 3^{-m}$, for each m ≥ 0.
4. (iv) $\mu(F_K) = 0$.
5. Let I_1 be the square with base the segment of E_1 with vertex at the origin. Then $F_K = 3(F_K$

$\cap I_1)$.

The General Dimension Features

The basic idea in most definitions of dimension is to find some measure Mδ(F) of the 'complexity of the structure' of a set F ⊂ R k at a size δ > 0, then see how Mδ(F) behaves as δ → 0. For instance, Mδ(F) could be any number Nδ(F) of balls of diameter δ needed to cover the set F. Now suppose that Mδ(F) obeys a power law of the form Mδ(F) = cδ−s . It will be seen that this is true for the number Nδ(F) if F is a fine curve, with s = 1, or for a smooth surface, with s = 2. Thus, it seems geometrically natural to regard s as the dimension of F, i.e., we define dim F := s. We can find s, i.e., dim F, from the power law by taking logs:

$\log M_\delta(F) = \log c + \log \delta^{-s} = \log c - s \log \delta \Rightarrow \dim F = \lim / \delta\to 0 \; {}^{\log M\delta(F)} / {}_{-\log \delta}$.

Of course, there may not be a perfect power law of Mδ (F) and this limit may not exist. We could then replace the lim by 'lim sup and lim inf' (a function that does not have a limit must 'oscillate' in some way, and these quantities give some idea of the upper and lower limits of the oscillations), which would give some sort of idea of the maximum and minimum complexity of F. We won't consider this idea further, but Falconer discusses it in the book. There are various ways of choosing the measure of complexity Mδ(F), and these lead to different definitions of dim F. Different definitions will lead to dimensions with slightly different properties. Below, one can consider the 'box-counting' dimension and the 'Hausdorff' dimension. The following is a list of what we would regard as desirable properties of dim F, and one can see below how many of these are satisfied by the two types of dimension.

The Dimensions Properties

1. Monotonicity E ⊂ F =⇒ dim E 6 dim F.
2. Finite stability dim(∪ m i=1Fi) = max{dim Fi : i = 1, . . . , m}.

3. Accountability dim(∪ ∞ i=1Fi) = sup{dim Fi : i = 1, 2, . . . }.
4. Sets which are open If F is open t dim F = k.
5. Smooth manifolds F = m (a manifold is a smooth curve or surface).
6. Geometric invariance If f : R k → R l is a 'rigid motion' then dim f(F) = dim F.
7. Lipschitz invariance If f : R k → R l is bi-Lipschitz then dim f(F) = dim F (Falconer, 2007).

For example

(i) Let I = [0, 1]. The intervals U_1 = (−1, 1), U_2 = (0, 2), are a 2-cover of I. (ii) Let F = I × {0} = {(x, 0) : x ∈ [0, 1]} ⊂ R^2 . The sets U_1 × {0}, U_2 × {0} ⊂ R^2 (where U_1, U_2 are the intervals in the previous example) again cover F. The balls $B_1(0)$, $B_1(1)$ also cover F. (iii) Let F = [0, 1) and let U_0 = (−1/2, 1/2), U_i = (1/2 − 1/2i, 1 − 1/2i), for i > 1, that is, U_1 = (0, 1/2), U_2 = (1/4, 3/4), U_3 = (1/3, 5/6), ... This is an (infinite) cover for F

Rather than count the covering in terms of sets of the δ diameter, the use of some different coverings can be applied. For instance, the set of all cubes of the form [m1δ,(m1 + 1)δ] × . . . × [mkδ,(mk + 1)δ] is called the δ-mesh (or δ-coordinate mesh) of R k . NB. the cubes in the δ-mesh have diameter δ √ k. Let N′ δ (F) be the number of cubes in the δ-mesh that intersect F. We could define the dimension by the limit in (3.1), but with Nδ(F) replaced by N′ δ (F). The following theorem shows that we obtain the same dimension.

Theorem

If either of the following limits exists, then both exist, and we have lim δ→0 log Nδ(F) − log δ = lim δ→0 log N′ δ (F) − log δ . 15 Proof. If {Ci} is a cover of F with δ-mesh cubes, then it is a general δ √ k-cover of F, so Nδ √ N (F) 6 N ′ δ (F). On the other hand, if {Ui} is a general δ-cover of F then any set Ui in this cover contains at most 3k δ- cubes (by the choice a cube mesh which contains some of the points of Ui , and then also taking the mesh cubes surrounding the first one). Thus, $N'_\delta(F) \leq 3^k N_\delta(F)$.

Fractals Properties

A set F ⊂ R k is usually called a fractal if it possesses the following properties.

- F is considered closed.
- ∂F = F.
- F is considered similar to itself.
- dimB F is not an integer.

Thus, all the examples we have discussed are fractals.

The function f : D ⊂ R k → R l is Lipschitz if there exists c > 0 so that |f (x) – f (y)| 6 c|x − y|, for x, y ∈ D points.

Note: A. bi-Lipschitz function f need not have an inverse f −1 , but if it does the condition (3.4) says that |f −1 (u) − f −1 (v)| 6 c|u − v|, for all u, v ∈ f(D) (putting u = f(x), v = f(y)), that is, both f and f −1 are Lipschitz with the same constant c. finite

. Let f : D → R l , F ⊂ D and δ > 0. (i) If f is Lipschitz and U1…,

UN is a δ-cover of F ⊂ R k then the sets f (U1), . ., f(UN) form a cδ-cover of f(F) ⊂ R l . (ii) If f is bi-Lipschitz and W1. . .

WN is a δ-cover of f(F) ⊂ R l then the sets f −1 (W1), . . . , f −1 (WN) form a cδ-cover of F ⊂ R k .

Proof: (i) Since the sets U1, . . . , UN cover F, the sets f(U1), . . . , f(UN) cover f(F). Now, for any x, y ∈ Uj , it follows from (3.3) that |f(x) − f(y)| 6 c|x − y| 6 cδ, so that diam(f(Uj)) 6 cδ. Hence, f(U1), . . . , f(UN) is a cδ-cover. (ii) The sets f −1 (W1), . . . , f −1 (WN) cover F. Now suppose that x, y ∈ f −1 (Wj).

Then it follows

x − y| 6 c|f(x) − f(y)| 6 cδ, so that diam(f −1 (Wj)) 6 cδ. Hence, f −1 (W1), . . . , f −1 (WN) is a cδ-cover.

Let f : D → R l , F ⊂ D and δ > 0. If f is bi-Lipschitz then Ncδ(f(F)) 6 Nδ(F) 6 Nc−1δ(f(F)), (3.5) Ncδ(F) 6 Nδ(f(F)) 6 Nc−1δ(F). (3.6) Proof. Let N = Nδ(F) and let U1, . . . , UN be a δ-cover of F. By part (i) of Lemma f(U1), . . . , f(UN) is a cδ-cover of f(F). Hence, the number of sets in the best possible cδ-cover of f(F) must be less than N, that is, Ncδ(f(F)) 6 Nδ(F), which is the first inequality in (3.5). Now let M = Nc−1δ(f(F)) and let W1, . . . , WM be a c −1 δ-cover of f(F). By part (ii) of Lemma 3.18, f −1 (W1), . . . , f −1 (WM) is a δ-cover of F, so Nδ(F) 6 Nc−1δ(F), which is the second inequality. The inequality now follows it (changing δ into cδ in the second inequality, which gives the first inequality in by changing δ into c −1 δ in the first inequality in and gives the second inequality.

Theorem

Let f : D → R l and F ⊂ D. If f : F → R l is bi-Lipschitz then dimB F exists iff dimB f(F) exists, in which case these dimensions are equal.

Proof: Suppose that dimB F exists. Hence, both the LHS and the RHS in the preceding inequality tend to dimB F, so the term in the middle also tends to dimB F, which proves that dimB f(F) exists and the dimensions are equal. 19 If dimB f(F) exists, we use inequality (3.5) in a similar manner to prove that dimB F exists and that the dimensions are equal. Theorem 3.20 shows that dimB is Lipschitz invariant, that is, the dimension of a set does not change if it is acted on by a bi-Lipschitz function. The next two theorems follow from this Lipschitz invariance.

Theorem

If $\Gamma 1$ is a C^1 curve in R^k, then $\dim_B \Gamma^1 = 1$. (ii) If $\Gamma 2$ is a C^1 surface in R^k, then $\dim_B \Gamma^2 = 2$. Proof. (i) It is known from differential geometry that $\Gamma 1$ can be split up into a union of smooth pieces $\Gamma 1 = \Gamma^1_1 \cup \cdots \cup \Gamma^1_M$, where for each piece Γ^1_j, there exists a function $f_j : B_1(0) \subset R^1 \to R^k$, such that f_j is C^1 and bi-Lipschitz. Since $\dim_B B_1(0) = 1$, it follows from Theorem 3.20 that $\dim_B \Gamma^1_j = 1$, and so by Theorem 3.13, $\dim_B \Gamma^1 = 1$. (ii) Similarly, $\Gamma 2 = \Gamma^2_1 \cup \cdots \cup \Gamma^2_M$, where for each Γ^2_j there exists $f_j : B_1(0) \subset R^2 \to R^k$, such that f_j is C^1 and bi-Lipschitz. Since $\dim_B B_1(0) = 2$, it follows from Theorem 3.20 that $\dim_B \Gamma^1_j = 2$, and so by the previous Theorem, $\dim_B \Gamma^1 = 2$.

Properties

The following is a summary of which of the desirable properties of a dimension listed based on the above examples as satisfied by dimB.

1. Monotonicity => True
2. Finite stability=>True
3. Countable stability=>False
4. Countable sets=>False
5. Open sets=>True
6. Smooth manifolds=>True
7. Lipschitz invariance=>True
8. Geometric invariance=>True

Comparison Of Dimh Versus Dimb

As was mentioned before, $\dim_H(F)$ is noted for all sets $F \subset R^k$, whereas $\dim_B(F)$ is only df is only found in bounded sets. There is also a need to consider the finer and more detailed coverings to achieve the inference in such a way that can make H^s_δ small.

Discrete Dynamical Systems

Let $D \subset R^k$, and $f : D \to D$. This section defines the function of a discrete dynamical system (DS). Intuitively, we think of n as time, and if we think of a point $x \in D$ as a 'starting point', then points $f_1(x)$, $f_2(x)$, . . . , are the only points to which x moves under the action of the dynamical system at each time step. This set of points is christened the "orbit" of x. The main problem: how does the sequence of points $f^n(x)$, $n = 1, 2, \ldots$, behave as $n \to \infty$, and how does this behavior depend on the 'starting point' $x \in D$? NB. the term 'discrete' is used here since we only look at the points at the orbit $f_1(x)$, $f_2(x)$, . . . , at discrete times $n = 1, 2, \ldots$. Continuous time dynamical systems can also be considered (in fact, solutions of differential equations give continuous time dynamical systems), but one cannot consider them here.

Notably, it is crucial to understand the difference between ‘p-periodic’ and ‘period p.’ The terminology is a bit confusing! The results of the next lemma are basically a restatement of the definition of a periodic point of f^p.

The points x = 0, 1 are the only fixed points. Similarly, the only p-periodic points are 0 and 1, so there are no periodic orbits with period ≥ 2.

This function could also be delineated by the single method $f_T(x) = 1 - 2|x - 1\ 2\ |$. The tent map has similar properties to the doubling function, but it is continuous. In particular, the tent map has lots of periodic points and orbits. One can discuss this function in a lot more detail below.

Graphical Analysis

In the 1-dimensional case, that is, when f : R → R, we can use the graph of f to help visualize the orbit of a given starting point x0 as follows:

- Draw the graph G of f, and also draw the straight line Δ given by y = x; NB. Fixed points occur when G intersects the line Δ.
- now choose a starting point x0 and draw the vertical line from x0 on the x-axis up to G (so we are at height y = f(x0));
- draw the horizontal line from here to the line Δ (we are now above the point f(x0) on the x-axis);
- draw the vertical line from here to G (so we are now at height y = f(f(x0)) = f 2 (x0));
- draw the horizontal line from here to the line Δ (we are now above the point f 2 (x0) on the x-axis);
- continue this process. The vertical lines we get lie above the orbit of x0. This graphical analysis can often give us a good idea of what the orbits will be like, even though it is not totally rigorous

Analysis

Almost all of what we do here will be in 1-dimension, so we won’t use phase portraits much since graphical analysis is more useful in 1-dimension.

Conclusion:

Therefore, it can be concluded that they are very helpful in modeling structures that have similar or progressively recurring small-scale patterns. A good example of such structures is the snowflakes. They are also used to describe random patterns such as galaxy development or crystal growth. The first person to ever use fractals was Benoit Mandelbrot in 1975. He based his definition of fractals on the Latin word which meant “broken” or fractured. There are several real-world fractal examples, such as the coastline of Great Britain, which has a jagged edge, and the leaves of plants, to mention a few. In mathematical terms, a fractal is defined as a set consisting of R k with a jagged-shaped structure. In order to clearly determine R k, some terms used need to be clarified to define what qualifies as a jagged structure.

1. If Γ1 is a C 1 curve in R k then dimB Γ 1 = 1.
2. (ii) If Γ2 is a C 1 surface in R k, then dimB Γ 2 = 2.

Proof: (i) It is known from differential geometry that Γ1 can be split up into a union of smooth pieces Γ1 = Γ1 1 ∪ · · · ∪ Γ 1 M, where for each piece Γ1 j there exists a function fj : B1(0) ⊂ R 1 → R k , such that fj is C 1 and bi-Lipschitz. Since dimB B1(0) = 1, it follows from Theorem 3.20 that dimB Γ 1 j = 1, and dimB Γ 1 = 1. (ii) Similarly, Γ2 = Γ2 1∪· · ·∪Γ 2 M, where for each Γ2 j there exists fj : B1(0) ⊂ R 2 → R k , such that fj is C 1 and bi-Lipschitz. Since dimB B1(0) = 2, it follows from Theorem 3.20 that dimB Γ 1 j = 2, and so by the previous Theorem, dimB Γ 1 = 2.

As was mentioned before, dimH(F) is noted for all sets F ⊂ R k, whereas dimB(F) is only df is only found in bounded sets. It is important to understand the difference between ‘p-periodic’ and ‘period p.’ The terminology is a bit confusing! The results of the next lemma are basically a restatement of the definition of a periodic point of f^{p}. where points x = 0, 1 are the only fixed points. Correspondingly, the only p-periodic points are 0 and 1, so there are no periodic orbits with a period ≥ 2.

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